

NORMS AND NORM-ATTAINABILITY OF NORMAL OPERATORS AND THEIR APPLICATIONS

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Abstract

The study of norms of operators forms a very important aspect in functional analysis, operator theory and its applications to economics, quantum chemistry and quantum computing amongst other fields. A lot of results have been obtained on norms of normal operators particularly by Kitanneh, Dragomir and Stacho among others. However, characterizations of norm-attainable operators have not been exhausted. The pending question that remains unanswered is: What are the necessary and sufficient conditions for normal operators to be norm-attainable? Moreover, what are the norms of these operators if the norm-attainability suffices? Therefore, in this paper, we present norms of operators in Hilbert spaces. We outline the theory of normal, self-adjoint and norm-attainable operators. The objectives of the study are: To determine norms of normal operators; To establish conditions for norm-attainability of normal operators; and to investigate norms of self-adjoint norm-attainable operators. The methodology involved the use of inner products, tensor products and some known mathematical inequalities like Cauchy-Schwarz inequality and the triangle inequality. The results obtained show that normaloid and normal operators are norm-attainable if there exist a unit vector x in the Hilbert space which is unique such that for any operator T , $\|Tx\| = \|T\|$. Furthermore, the operators are norm-attainable if they are self-adjoint. These results concur with the results of Qui Bao Gao for compact operators when the Hilbert space is taken to be infinite dimensional and complex. In conclusion, the results obtained are useful in quantum computing in generating quantum bit. In genetics the results are useful in the determination of DNA results of parents and offspring.

Key words: Hilbert space, normality, norm-attainability, self-adjoint operators, tensor products

1.0 Introduction

Many researchers and mathematicians have done a lot of studies on Hilbert space operators and their properties. According to Dragomir and Moslehian (2008) the study of norms of operators forms a very important aspect in functional analysis, operator theory and its applications to economics, quantum chemistry and quantum computing amongst other fields. A lot of results have been obtained on norms of normal operators particularly by Dragomir (2007) among others. However, characterizations of norm-attainable operators have not been exhausted. The pending question that remains unanswered is: What are the necessary and sufficient conditions for normal operators to be norm-attainable? Moreover, what are the norms of these operators if the norm-attainability suffices? Therefore, in this paper, we present norms of operators in Hilbert spaces. We outline the theory of normal, self-adjoint and norm-attainable operators.

Let H be a complex Hilbert space with the usual norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $B(H)$ denote the algebra of all bounded linear operators on H , $r(S)$ will also denote the numerical radius.

2.0 Preliminaries

Here we start by defining some key terms that are useful in the sequel.

Definition 2.1 An operator $S \in B(H)$ is normaloid if

$$\|S\| = r(S).$$

Definition 2.2 The numerical radius of an operator is $S \in B(H)$ defined by $r(S) = \sup\{|\lambda|: \lambda \in W(S)\}$ where $W(S)$ is the numerical range of S given by

$$W(S) = \{\langle Sx, x \rangle: x \in H, \|x\| = 1\}.$$

Definition 2.3 Let (V, \mathbb{K}) be an inner product space. Then $\forall x, y \in V$, x and y are said to be orthonormal if $\langle x, y \rangle = 0$ and $\|x\| = \|y\| = 1$. An orthonormal set of all vectors of the form x and y which form a basis called an orthonormal basis.

3.0 Main Results

Here we establish some new results concerning reverse inequalities for normaloid operators and norm-attainability Dragomir(2008)

Theorem 4.1 Let $S \in B(H)$ be a positive normal and normaloid operator. If $\alpha \in \mathbb{C} \setminus \{0\}$ and $m > 0$ are such that

$$\|S - \alpha x\| \leq m. \tag{1}$$

Then,

$$(0 \leq) \|S\| - r(S) \leq \frac{1}{2} \frac{m^2}{|\alpha|}. \text{ Moreover, } S \text{ is norm-attainable} \tag{2}$$

Proof. We follow a similar argument to the one from Dragomir(2007a)

Without loss of generality, assume that S is normal. For $x \in H$ with $\|x\| = 1$, from (1) we have

$$\|S - \alpha x\|^2 \leq m^2.$$

Which upon expanding gives

$$\begin{aligned} \|S - \alpha x\|^2 &= \langle S - \alpha x, S - \alpha x \rangle \\ &= \langle Sx, Sx \rangle - \langle Sx, \alpha x \rangle - \langle \alpha x - Sx \rangle + \langle \alpha x, \alpha x \rangle \\ &= \|Sx\|^2 - 2\text{Re}[\bar{\alpha}\langle Sx, x \rangle] + |\alpha|^2. \end{aligned}$$

So,

$$\|Sx\|^2 - 2\text{Re}[\bar{\alpha}\langle Sx, x \rangle] + |\alpha|^2 \leq m^2$$

$$\|Sx\|^2 + |\alpha|^2 \leq 2|\alpha|\langle Sx, x \rangle + m^2 \tag{3}$$

Taking the supremum over $x \in H, \|x\| = 1$ in (3) we obtain

$$\|S\|^2 + |\alpha|^2 \leq 2|\alpha| \sup\{|\langle Sx, x \rangle|\} + m^2$$

But, $r(S) = \sup\{|\langle Sx, x \rangle| : x \in H, \|x\| = 1\}$.

Thus

$$\|S\|^2 + |\alpha|^2 \leq 2|\alpha|r(S) + m^2 \tag{4}$$

$$\text{But, } \|S\|^2 + |\alpha|^2 \geq 2|\alpha|\|S\| \tag{5}$$

Hence from Equations (4) and (5) we obtain the required equation (2).

Corollary 4.2 Let S, α and m be as in Theorem (4.1) and

$$|\alpha| - r(S) \geq n \tag{6}$$

For some $n \geq 0$, then

$$(0 \leq) \|S\|^2 - r^2(S) \leq m^2 - n^2 \tag{7}$$

Proof. From Equation (4) i.e.

$$\|S\|^2 + |\alpha|^2 \leq 2|\alpha|r(S) + m^2. \tag{8}$$

Squaring Equation (6) both sides we have

$$|\alpha|^2 + r^2(S) - n^2 \geq 2|\alpha|r(S). \tag{9}$$

Substituting Equation (9) into Equation (8) we obtain

$$(0 \leq) \|S\|^2 - r^2(S) \leq m^2 - n^2.$$

Corollary 4.3 Let S, α and m be as in Theorem (4.1). Consider an orthonormal sequence x_p and an orthonormal basis $\{e_p\}_{p \in \mathbb{N}}$ in the domain of S then

$$\lim_{p \rightarrow \infty} \left| |e_p| - r(S) \right| \geq m^{p-1}.$$

Proof. Follows analogously from [Dragomir(2006), Theorem 3].

Theorem 4.4 Let $S = (x_1 \otimes y_1)$ and $T = (x_2 \otimes y_2)$. Then

$$||x^p| - r(S \otimes T)| \geq m^2.$$

Proof. Without loss of generality let T be norm-attainable then $\|Tx\| = \|T\|$. By corollary 4.1 and Corollary 4.3 we invoke the completely bounded norm by Dragomir(2007b) and the proof is complete.

4.0 Applications

The results obtained show that normal operators are norm-attainable if there exist a unit vector x in the Hilbert space which is unique such that for any operator T , $\|Tx\| = \|T\|$. Furthermore, the operators are norm-attainable if they are self-adjoint. These results concur with the results of Qui Bao Gao for compact operators when the Hilbert space is taken to be infinite dimensional and complex. There are several applications of these results, for example, in quantum computing in generating quantum bit. In genetics the results are useful in the determination of DNA results of parents and offspring.

5.0 Conclusion

In conclusion, in this paper we have obtained norm estimates and norm-attainability conditions for Hilbert space operators.

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