ON THE SPECTRUM OF A SPECIAL NORLUND MEANS

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Abstract
In a series of papers, some authors have previously investigated and determined the spectrum of weighted mean matrices considered as bounded operators on various sequence spaces. Coskun (2003) determined the set of eigenvalues of a special Norlund matrix as a bounded operator over some sequence spaces. In 2010, Akanga, Mwathi, and Wali, determined the spectrum of a special Norlund matrix as a bounded operator on c_0. It is evident that no research has been done on the spectrum of a general Norlund means. In this paper the spectrum of a special Norlund matrix as a bounded operator on the sequence space c is determined. This is achieved by applying Banach space theorems of functional analysis as well as summability methods of summability theory. In which case it is shown that the spectrum comprises the set

$$\left\{\lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| \le \frac{1}{2}\right\}$$

Key words: Boundedness, operator, spectrum, norm, convergence

Notations: $\not\subset; \sigma(T); \Re; c; c_0; \ell_p(0)\ (0 < p < \infty); \ell_\infty; \|T\|; \theta; A^t$ **,** will denote the set of all

Complex numbers; the spectrum of T; the set all real numbers; the space of all convergent sequences; the space of null sequences; the space of sequences such that

$\sum_{k=0}^{\infty} |x_k|^p < \infty$; the space of bounded sequences; the norm of T; the zero sequence; the transpose of A.

1.0 Introduction

Functional analysis is often referred to as the queen of applied sciences. Indeed it finds a lot of applications through summability theory. Broadly speaking, summability is the theory of assignment of limits,which is fundamental in analysis. Spectrum of an operator plays a crucial role in development of a Tauberian theory of an operator, i.e., the determination of the limit of a convergent sequence or series from the convergence of its matrix transform.It also plays a central role in Fourier analysis, and analytic continuation of functions.

1.1 Classical Summability

The central problem in summabitiy is to find means of assigning a limit to a divergent sequence or sum to a divergent series. In such a way that the sequence or series can be manipulated as though it converges, (Ruckle,1981), pp. 159-161. The commonest means of summing a divergent series or sequence is that of using an infinite matrix of complex numbers, or by a power series.

Definition 1.1.1 (sequence to sequence transformation)

Let A = (a_{nk}) n, k = 0, 1, 2,…, be an infinite matrix of complex numbers. Given a sequence x = $(x_k)_{k=0}^{\infty}$, define

$$y_n = \sum_{k=0}^{\infty} a_{nk}, n = 0,1,2,... \quad \text{.....(1.1)}$$

If the series (1.1), converges for all n, then we call the sequence $(y_n)_{n=0}^{\infty}$, the A – transform of the sequence $(x_k)_{k=0}^{\infty}$. If further, $y_n \to a$, as $n \to \infty$, we say that $(x_k)_{k=0}^{\infty}$ is summable A to a.

Examples of sequence to sequence transformations:

Example 1.1.1 (Cesaro matix means)

The matrix A = (a_{nk}), where

$$a_{nk} = \begin{cases} \frac{1}{1+n}, 0 \le k \le n \\ 0, k > n \end{cases} \quad \text{.....(1.2)}$$

Is called the Cesaro matrix of order 1, denoted by (C, 1), or, C_1. This matrix sums (1, 0, 1, 0,…) to $\frac{1}{2}$.

Example 1.1.2 (Norlund means)

The transformation given by

$$y_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} x_k, \text{ n =0, 1, 2,} \quad \text{.....(1.3)}$$

Where $P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0$, is called Norlund means, and is denoted by

$$a_{nk} = \begin{cases} \frac{p_{n-k}}{P_n}, 0 \le k \le n \\ 0, k > n \end{cases} \quad \text{.....(1.4)}$$

In matrix (1.4), if $p_0 = 1, p_1 = -2, p_2 = p_3 = p_4 = ... = 0$,then A = (a_{nk}) Transforms the unbounded sequence (1,2,4,8,16,…) to zero. If $p_n = 1$, for each n = 0, 1, 2, …, then $(a_{nk}) = (C,1)$. If in matrix (1.4), $p_0 = 1, p_1 = 1, p_2 = p_3 = p_4 = ... = 0$, then $(a_{nk}) = (q_{nk})$ - the Q matrix, given by

$$q_{nk} = \begin{cases} 1, n = k = 0 \\ \frac{1}{2}, n-1 \le k \le n \\ 0, otherwise \end{cases} \quad \text{.....(1.5)}$$

Which is the matrix of interest in this paper.

1.2 Some General Results in Classical Summability

Definition 1.2.1 (Regular method, Conservative method)

Let A = (a_{nk}), n =0,1,2,..., be an infinite matrix of complex numbers.

(i) If the A – transform of any convergent sequence of complex numbers exists and converges, then A is called a conservative method. We, then write
A $\in (c,c)$.

(ii) If A is conservative and preserves limits, then A is called regular. We, then write
A $\in (c,c,P)$

Theorem 1.2.1 (Kojima – Shur)

A $\in (c,c)$ if and only if

(i) $a_{nk} \to a_k, n \to \infty$, for each $k \geq 0$;

$(ii)\sum_{k=0}^{\infty} a_{nk} \to a, n \to \infty$;

$(iii)\sup_{n\geq 0}\left\{\sum_{k=0}^{\infty}|a_{nk}|\right\} < \infty$

(Maddox, 1970), pp. 166 – 167

Theorem 1.2.2

A $\in (\ell_1, \ell_1)$ iff

$(i)\sum_{n=0}^{\infty}|a_{nk}| < \infty$, for each k;

$(ii)\sup_k \sum_{n=0}^{\infty}|a_{nk}| < \infty$

(Limaye, 1996), pp.88 – 90

1.3 Some Results from Functional Analysis

Definition 1.3.1 (Norm)

A norm on a real (or complex) vector space X is a real – valued function on X, whose value at an $x \in X$ is denoted by $\|x\|$, and which has the properties:

(i) $\|x\| \geq 0$

(ii) $\|x = 0\|$, iff x = 0

(iii) $\|\alpha x\| = |\alpha|\|x\|$

(iv) $\|x + y\| \leq \|x\| + \|y\|$, where x, y $\in X$, and $\alpha \in \mathrm{K}$.

(Kreyszig, 1978), page 59.

Example 1.3.1

$c_0; c; \ell_p (p \geq 1); \ell_\infty$ are all normed linear spaces. Their norms are as follows:

$c_0; c; \ell_\infty$ have the same natural norm, namely $\|x\| =$ sup $_{n\geq 0}\{|x_n|\}$; $\ell_p (1 \leq p < \infty)$,
Has natural norm

$$\|x\| = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}}$$

Definition 1.3.2 (adjoint operator T *)

The adjoint operator T * of a linear operator $T \in B(X, Y)$ is the mapping from Y^* to X^* defined by

$$T^* of = foT, f \in Y^*$$

Theorem 1.3.1

T^* is linear and bounded. Morever, $\|T^*\| = \|T\|$

(Dunford and Schwartz, 1957), page. 478.

Definition 1.3.3 (Resolvent set $\rho(T)$, spectrum $\sigma(T)$)

Let X be a non empty Banach space and suppose that $T : X \to X$. The resolvent set $\rho(T)$ of T is the set of complex numbers, λ, for which $(T - \lambda I)^{-1}$ exists as a bounded operator with domain X. The spectrum $\sigma(T)$ of T is the compliment of $\rho(T)$ in $\not\subset$.

Theorem 1.3.2

Let $T \in (X)$, where X is any Banach space, the spectrum of T^* is identical with the spectrum of T.

(Goldberg, 1966), page 71

2.0 The Spectrum of Q Operator on c (Main Results)

In this section, we determine the spectrum of Q matrix as an operator on c.

Corollary 2.1 $Q \in B(c)$, moreover

$$\|Q\|_c = \|Q^*\|_{\ell_1} = 1$$

Proof: The validity of parts (i), (ii), and (iii) of theorem (1.2.1). From matrix (1.5) it is evident that

$\lim_{n\to\infty} q_{nk} = 0,$ for each $k \geq 0$ ....(2.1)

and $\sum_{k=0}^{\infty} q_{nk} = \sum_{k=0}^{n} q_{nk} = 1$, for each n ....(2.2)

so that $\lim_{n\to\infty} q_{nk} = 1$ .... (2.3)

Hence,

$\sup_{n\geq 0} \left\{ \sum_{k=0}^{\infty} |a_{nk}| \right\} = 1 < \infty$ ....(2.4)

Therefore $Q \in B(c)$, moreover $\|Q\|_c = 1$

Theorem 2.1 Let $T : c \to c$ be a linear map and define $T^* : c^* \to c^*$,i.e., $T^* : \ell_1 \to \ell_1$, by $T^*(g) = goT$, $g \in c^* \equiv \ell_1$. Then both T and T^* must be given by a matrix. Moreover $T^* : \ell_1 \to \ell_1$ is given by the matrix

$$A^* = T^* = \begin{pmatrix} \chi(\lim_A) & (v_n)_0^\infty \\ (a_k)_0^\infty & A^t \end{pmatrix} \quad \text{..........} \quad (2.5)$$

where,

$$\chi(\lim_A) = \lim_A(\delta) - \sum_{k=0}^{\infty} \lim_A \delta^k;$$

$$v_n = \chi(P_n o T);$$

$$a_{nk} = P_n\left(T\left(\delta^k\right)\right) = \left(T\left(\delta^k\right)\right)_n;$$

$$a_k = \lim_{n\to\infty} a_{nk};$$

$$\delta = (1,1,1,\ldots);\ \delta^k = \left(\delta_n^k\right)_{n=0}^{\infty} = (0,0,\ldots,0,1,\ldots) \quad \text{..........} \quad (2.6)$$

(Wilansky, 1984), page 267

Corollary 2.2 Let $Q : c \to c$, then $Q^* \in B(\ell_1)$, moreover

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 1 & \frac{1}{2} & 0 & 0 & . & . & . \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & . & . & . \\ & & . & . & . & & & \end{pmatrix} \quad \text{..........} \quad (2.7)$$

Proof: By theorem (2.1) $Q^* \in B(\ell_1)$, moreover

$$Q^* = \begin{pmatrix} \chi(\lim_Q) & (v_n)_0^\infty \\ (a_k)_0^\infty & Q^t \end{pmatrix} \quad \text{..........} \quad (2.8)$$

But for Q matrix, $v_n = \theta$, and $(a_k)_0^\infty = \theta$, since $\lim_{n\to\infty} q_{nk} = 0, \forall k \geq 0$;
and $\chi(\lim_Q) = 1$, hence the result.

Theorem 2.2: $Q \in B(c)$**,** has only one eigenvalue, $\lambda = 1$, corresponding to the Vector $x = \delta = (1,1,1,\ldots)$

Proof: suppose, $Qx = \lambda x$, $x \neq \theta$ in c and $\lambda \in \not\subset$. Then solving the system, we If x_0 is the first non- zero entry of x, then $\lambda = 1$. But $\lambda = 1$, implies that $x_0 = x_1 = x_2 = \ldots = x_n = \ldots \in c$. Hence, $\lambda = 1$ is an eigenvalue of Q. When

$x_{n+1}, n = 0,1,2,\ldots$ is the first non-zero entry of x, then $\lambda = \frac{1}{2}$. Solving the system with $\lambda = \frac{1}{2}$, results in $x_n = 0, n = 0,1,2,\ldots$, a contradiction. Hence

$\lambda = \frac{1}{2}$ cannot be an eigenvalue of $Q \in B(c)$. Hence the result.

Theorem 2.3: The eigenvalues of $Q \in B(\ell_1)$ form the set

$$\left\{\lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| < \frac{1}{2}\right\} \cup \{1\}$$

Proof: Suppose $Q^* = \lambda x, x \neq \theta$ and $\lambda \in \not\subset$
Then we have

$x_0 = \lambda x_0, x_1 + \frac{1}{2}x_2 = \lambda x_1$, and $\frac{1}{2}(x_n + x_{n+1}) = \lambda x_n, n \geq 2$ .... (2.9)

Solving system (2.9) with $\lambda = 1, x_0 \neq 0$ gives the vectors

$x^1(x_0, 0,0,0,\ldots), x^2 = (x_0, x_1, 0,0,\ldots) \in \ell_1$. Hence $\lambda = 1$

is an eigenvalue of $Q^* \in B(\ell_1)$. Similarly slving the system for $x_n, n \geq 2$ in terms of x_1 yield

$$x_n = 2^{n-1}\lambda^{n-1}\left(1 - \frac{1}{\lambda}\right)\left(1 - \frac{1}{2\lambda}\right)^{n-2} x_1, n \geq 2 \quad \text{... (2.10)}$$

By the ratio theorem the vector is in ℓ_1 iff

$$\left|\lambda - \frac{1}{2}\right| < \frac{1}{2}$$

Hence, the theorem.

Theorem 2.4

$$\sigma(Q) = \left\{\lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$$

Proof: Solving the system $(Q - \lambda I)x = y$, for x in terms of y, yields

$$x_n = -\frac{1}{2^n \lambda^{n+1}\left(1 - \frac{1}{2\lambda}\right)^n \left(1 - \frac{1}{\lambda}\right)} y_0 - \frac{1}{2^{n-1}\lambda^n \left(1 - \frac{1}{2\lambda}\right)^n} y_1 - \frac{1}{2^{n-k}\lambda^{n-k+1}\left(1 - \frac{1}{2\lambda}\right)^{n-k+1}} y_k, 0 \leq k \leq n \quad \text{.........(2.11)}$$

Which, yields the matrix

$$M=(m_{nk})=\begin{cases}-\dfrac{1}{2^n\lambda^{n+1}\left(1-\dfrac{1}{2\lambda}\right)^n\left(1-\dfrac{1}{\lambda}\right)}, k=0\\ -\dfrac{1}{2^{n-k}\lambda^{n-k+1}\left(1-\dfrac{1}{2\lambda}\right)^{n-k+1}}, 1\le k\le n\\ 0, k>n\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.12)}$$

Note that

$$(Q-\lambda I)=M^{-1}=\begin{cases}1-\lambda, k=n=0\\ \dfrac{1}{2}, k=n-1\\ 0, otherwise\end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(2.13)}$$

Also note that

$$MM^{-1}=M^{-1}M=I$$

We now check that $M\in B(c)$

Columns of M are null, provided $\left|\dfrac{(n+1)thterm}{nthterm}\right|<1$ ……………………………………………(2.14)

Applying condition (2.14) to matrix (2.12), gives columns of M

Are null provided, $\lambda\in\not\subset$ is such that $\left|\lambda-\dfrac{1}{2}\right|>\dfrac{1}{2}$.

Remark 2.1 For any matrix $A=(a_{nk})$, if $\lim_n a_{nk}=0,\forall k\ge 0$, then

$\sup_{n\ge 0}\sum_{k=0}^{\infty}|a_{nk}|<\infty$ (Maddox, 1970), page 164, or (Reade, 1985),page 266.

Summing absolutely along the nth row of matrix M, gives

$$\sum_{k=0}^{\infty}|m_{nk}|=\left|\frac{-1}{2^n\lambda^{n+1}\left(1-\frac{1}{2\lambda}\right)^n\left(1-\frac{1}{\lambda}\right)}\right|+\sum_{k=1}^{n}\left|\frac{-1}{2^{n-k}\lambda^{n-k+1}\left(1-\frac{1}{2\lambda}\right)^{n-k+1}}\right|=\varepsilon_n \text{, say, } n\ge 0 \quad \text{(2.15)}$$

By remark (2.1) sup $_n$ $\{\varepsilon_n\} < \infty$, provided $\lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Which deals with

Parts (i), and (iii) of theorem (1.2.1)

Now, $\left|\sum_{k=0}^{n} m_{nk}\right| \leq \sum_{k=0}^{n} |m_{nk}| = \varepsilon_n, n \geq 0$ (2.16)

Which implies that, $\lim_n \sum_{k=0}^{\infty} m_{nk}$, exists provided, $\lambda \in \not\subset$ is such that $\left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$.
Hence part (ii) of theorem (1.2.1)

So that $M = (Q - I\lambda)^{-1} \in B(c), \forall \lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| > \frac{1}{2}$. Hence,

$M = (Q - I\lambda)^{-1} \notin B(c), \forall \lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}$. Hence the theorem.

3.0 Conclusion

In this paper the spectrum of $Q \in B(c)$ has been determined as $\sigma(Q) = \left\{\lambda \in \not\subset : \left|\lambda - \frac{1}{2}\right| \leq \frac{1}{2}\right\}$. It has also been shown that $Q \in B(c)$ has only one eigenvalue, $\lambda = 1$ corresponding to the vector $x = \delta = (1,1,1,\ldots)$.

Acknowledgement
I would like to acknowledge the late Prof John Ingutia Okutoyi. Who initiated me in the exiting world of summability theory.

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