SOME RESULTS ON ANTI-INVARIANT MAXIMAL SPACELIKE SUBMANIFOLDS OF AN INDEFINITE COMPLEX SPACE FORM

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ABSTRACT
This paper looks into the geometry of an n-dimensional anti-invariant maximal spacelike submanifold $M$ immersed in an indefinite complex space form $\mathcal{M}(c), c \neq 0$. Let $M$ be an n-dimensional compact anti-invariant maximal spacelike submanifold of $\mathcal{M}^{n+p}_{\rho}(c), c \neq 0$. Then we show that either $M$ is totally geodesic or

\[ S = \frac{(n+1)(n+2p)}{4(2n+4p-1)}c \]

or at some point of $M$,

\[ S > \frac{(n+1)(n+2p)}{4(2n+4p-1)}c. \]

Key words: Anti-invariant submanifold, complex space form
1.0 INTRODUCTION
Among all submanifolds of a Kaehler manifold, there are two classes the class of anti-invariant submanifolds and that of holomorphic submanifolds. A submanifold of a Kaehler manifold is called an anti-invariant (resp. holomorphic) if each tangent space of the submanifold is mapped into the normal space (resp. itself) by the almost complex structure of the Kaehler manifold (Chen et al., 1974). A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form. Let \( \bar{M}(c), c \neq 0 \) be an indefinite complex space form of holomorphic sectional curvature c, complex dimension \( (n+p) \), \( p \neq 0 \) and index 2p. Let M be an n-dimensional anti-invariant maximal spacelike submanifold isometrically immersed in \( \bar{M}(c) \). We call M a spacelike submanifold if the induced metric on M from that of the ambient space is positive definite. Let J be the almost complex structure of \( \bar{M}(c) \). An n-dimensional Riemannian manifold M isometrically immersed in \( \bar{M}(c) \) is called an anti-invariant submanifold of if each tangent space of M is mapped into the normal space by the almost complex structure J. Let h be the second fundamental form of M in \( \bar{M}(c) \) and denote by S the square of the length of the second fundamental form h.

Our main result is:

\[ \text{Theorem.} \]

Let M be an n-dimensional compact anti-invariant maximal spacelike submanifold of \( \bar{M}_{n+p}(c), c \neq 0 \). Then either M is totally geodesic or

\[ S = \frac{(n+1)(n+2p)}{4(2n+4p-1)} c \]

or at some point of M,

\[ S = \frac{(n+1)(n+2p)}{4(2n+4p-1)} c \]

2.0 LOCAL FORMULAE
We choose a local field of orthonormal frames:
\[ \{e_1,...,e_n; e_{n+1},...,e_{n+p}; e_{n+p} = Je_1;...,e_n = Je_n; e_{(n+1)p} = Je_{n+1},...,e_{(n+p)p} = Je_{n+p}\} \]

in \( \bar{M}_{n+p}(c) \) such that restricted to M, the vectors \( \{e_1,...,e_n\} \) are tangent to M and the rest are normal to M. With respect to this frame field of \( \bar{M}_{n+p}(c) \) let
\[ w^1,...,w^n; w^{n+1},...,w^{n+p}; w^1;...,w^n; w^{(n+1)p};...,w^{(n+p)p} \]
be the field of dual frames.

Unless otherwise stated, we shall make use of the following convention on the ranges of indices: \( 1 \leq A, B, C, D \leq n + p \); \( 1 \leq i, j, k, l, m \leq n \); \( n + 1 \leq a, b, c \leq n + p \); and when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Besides
\[ e_i = g(e_i, e_i) = g(Je_i, Je_i) = 1 \]
when \( 1 \leq i \leq n \).
\[ e_a = g(e_a, e_a) = g(Je_a, Je_a) = -1 \quad \text{when} \quad n + 1 \leq a \leq n + p. \]

Then the structure equations of \( \tilde{M}_{n+p}^c(c, c \neq 0) \) are:

\[ d\omega^A + \sum e_B \omega^A_B \wedge \omega^B = 0, \quad \omega^A_B + \omega^B_A = 0, \quad \omega^*_j = \omega^*_j, \quad \omega^*_j = \omega^*_j, \]

\[ d\omega^A_B + \sum e_C \omega^A_C \wedge \omega^B_C = \frac{1}{2} \sum e_D \omega^A_D \tilde{R}^A_{BCD} \omega^B \wedge \omega^D, \]

\[ \tilde{R}^A_{BCD} = \frac{c}{4} e_c e_D (d_{AC} d_{BD} - d_{AD} d_{BC} + J_{AC} J_{BD} - J_{AD} J_{BC} + 2 J_{AB} J_{CD}) \]

where \( \tilde{R}^A_{BCD} \) denote the components of the curvature tensor \( \tilde{R} \) on. Restricting these forms to \( M \) we have:

\[ \omega^A = 0, \quad \omega^*_j = \sum_i h^a_j \omega^i, \quad h^a_j = h^a_j, \quad d\omega^j = -\sum \omega^i \wedge \omega^j, \]

\[ \omega^*_j + \omega^j = 0, \quad d\omega^i_j = -\sum \omega^i_k \wedge \omega^j_k + \frac{1}{2} \sum R^i_{jk} \omega^k \wedge \omega^j, \]

\[ R^i_{jkl} = \tilde{R}^i_{jkl} - \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}), \quad d\omega^a = -\sum \omega^b_i \wedge \omega^b, \]

\[ d\omega^a_b = -\sum e_c \omega^a_c \wedge \omega^b + \frac{1}{2} R^a_{bij} \omega^i \wedge \omega^j, \]

\[ R^a_{bij} = \sum_k (h^a_{ik} h^b_{kj} - h^a_{ij} h^b_{kj}) \]

(2.1)

From the condition on the dimensions of \( M \) and \( \tilde{M}_{n+p}^c(c) \), it follows that \( e^*_1, \ldots, e^*_n \) is a frame for \( T^\perp(M) \). Noticing this, we see that

\[ R^i_{jkl} = \frac{c}{4} (d^i_{jk} - d^i_{kj}) - \sum_a (h^a_{ik} h^a_{jl} - h^a_{il} h^a_{jk}) \]

(2.2)

\[ H = \frac{1}{n} \sqrt{\sum_a \left( \sum_i h^a_{ij} \right)^2} \]

We call the mean curvature of \( M \) and the square of the length of the second fundamental form. If \( H \) is identically zero, then \( M \) is said to be maximal. \( M \) is totally geodesic if \( h = 0 \).

From (2.2), we have the Ricci tensor \( R^i_j \) given by

\[ R^i_j = \sum_k R^i_{jk} = \frac{(n-1)}{4} c d^i_j + \sum_{ak} h^a_{ik} h^a_{kj} \]

(2.3)

Thus the Ricci curvature \( R \) is;
\[ R = R_i^j = \frac{c}{4} (n-1) + S \]

..................................................................................(2.4)

From (2.3) the scalar curvature \( r \) is given by

\[ r = \sum_j R_j^i = \frac{n(n-1)}{4} c + S \]

..................................................................................(2.5)

Let \( h_{ijk}^a \) denote the covariant derivative of \( h_{ij}^a \). Then we define \( h_{ijk}^a \) by

\[ \sum_k h_{ijk}^a w^k = dh_{ijk}^a + \sum_k h_{ikj}^a w^k + \sum_k h_{ijk}^a w^k + \sum_k h_{ijk}^a w^k \]

..................................................................................(2.6)

and \( h_{ijk}^a = h_{ijk}^a \). Taking the exterior derivative of (2.6) we define the second covariant derivative of \( h_{ij}^a \) by

\[ \sum_i h_{i[jk}^a w^i = dh_{ijk}^a + \sum_i h_{ij[k}^a w^i + \sum_i h_{ijk}^a w^i + \sum_i h_{ijk}^a w^i \]

(2.7) Using (2.7), we obtain the Ricci formula;

\[ h_{ijkl}^a - h_{ij[kl}^a = \sum_m h_{mk}^a R_{ijkl}^m + \sum_m h_{kn}^a R_{ijkl}^m + \sum_b h_{ij[k}^a R_{ijkl}^a \]

..................................................................................(2.8)

The Laplacian of the second fundamental form is defined as

\[ \Delta = \sum \nabla \nabla \]

where \( H_{ij} \) is the second covariant derivative of \( H \).

Therefore,

\[ \Delta h_{ij}^a = nH_{ij} + \frac{c}{4} (n+1) \sum h_{ij}^a - \sum h_{mij}^a h_{njk}^b h_{ik}^b + \sum h_{mij}^a h_{njk}^b h_{ik}^b - \sum h_{km}^a h_{mj}^b h_{ik}^b \]

(2.9)

where \( H_{ij} \) is the second covariant derivative of \( H \).

For M maximal in \( \operatorname{M}^{n+p} \), (2.9) becomes,

\[ \Delta h_{ij}^a = \frac{c}{4} (n+1) \sum h_{ij}^a - \sum h_{mij}^a h_{njk}^b h_{ik}^b - \sum h_{km}^a h_{mj}^b h_{ik}^b \]

(2.10)
From \( \frac{1}{2} \Delta \sum_{a_{ij}} (h_{ij}^a)^2 = \sum_{a_{ijk}} (h_{ijk}^a)^2 + \sum_{a_{ij}} h_{ij}^a \Delta h_{ij}^a \) we obtain,

\[
\frac{1}{2} \Delta \sum_{a_{ij}} (h_{ij}^a)^2 = \sum_{a_{ijk}} (h_{ijk}^a)^2 + \frac{c}{4} (n+1) \sum_{a_{ij}} (h_{ij}^a)^2 - \sum_{abijkl} h_{ij}^a h_{kl}^a h_{kl}^b h_{ij}^b + \sum_{abijkl} (h_{ij}^a h_{kl}^b - h_{kl}^a h_{ij}^b) (h_{ij}^b h_{kl}^a - h_{kl}^b h_{ij}^a)
\]

..............................................................(2.11)

For each a let \( H_a \) denote the symmetric matrix \( (h_{ij}^a) \). Then (2.11) can be written as

\[
\frac{1}{2} \Delta \sum_{a_{ij}} (h_{ij}^a)^2 = \sum_{a_{ijk}} (h_{ijk}^a)^2 + \frac{c}{4} (n+1) \sum_{a_{ij}} (h_{ij}^a)^2 - \sum_{ab} (tr H_a H_b)^2 + \sum_{ab} tr \left( H_a H_b - H_b H_a \right)^2
\]

..............................................................(2.12)

where \( tr H_a H_b \) denotes the trace of the matrix \( H_a H_b \).

In the sequel, we need the following lemma proved in (Chern et al., 1970) by S. S. Chern, M. do Carmo and S. Kobayashi.

**Lemma 2.1:**

Let \( A \) and \( B \) be symmetric nxn-matrices. Then, \(- \mathbf{F} \left( B - B^T \right) \leq 2TrA^2TrB^2\) and equality holds for non-zero matrices \( A \) and \( B \) if and only if \( A \) and \( B \) can be transformed by an orthogonal matrix simultaneously into scalar multiples of \( \overline{A} \) and \( \overline{B} \) respectively, where

\[
\overline{A} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \overline{B} = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Moreover, if \( A_1, A_2, A_3 \) are three symmetric nxn-matrices such that

\[- \mathbf{F} \left( A_a A_b - A_b A_a \right)^2 = 2TrA_a^2TrA_b^2, \quad 1 \leq a, b \leq 3, \quad a \neq b,\]

then at least one of the matrices \( A_a \) must be zero.

\[
S_{ab} = \sum_{a_{ij}} h_{ij}^a h_{ij}^b
\]

Let \( S_{ab} \). Then \((n+2p) \times (n+2p)\)-matrix \( \left( S_{ab} \right) \) is symmetric and can be assumed to be diagonal for a suitable choice of \( e_{n+1}, \ldots, e_{n+p} \). Setting

\[
S_a = S_{aa} = tr H_a^2 \quad \text{and} \quad S = \sum_a S_a
\]

, equation (2.12) reduces to
\[
\frac{1}{2} \Delta S = \sum_{ijk} \left(h_{ijk}^a\right)^2 + \frac{c}{4} (n+1)S - \sum_{ab} \left(trH_aH_b\right)^2 + \sum_{ab} tr\left(H_aH_b - H_bH_a\right)^2
\]  
(2.13)

On the other hand, using Lemma 2.1 we have,

\[
\frac{c}{4} (n+1)S - \sum_{ab} \left(trH_aH_b\right)^2 + \sum_{ab} tr\left(H_aH_b - H_bH_a\right)^2 \geq \frac{c}{4} (n+1)S - \sum_a S_a^2 - 2\sum_{ab} S_aS_b
\]

\[
= \left(\frac{(1-2n-4p)}{n+2p}\right)S + \frac{c}{4} (n+1)S + \frac{1}{(n+2p)}\sum_{a>b} \left(S_a - S_b\right)^2
\]

(2.14)

which, together with (2.13), implies that

\[
\frac{1}{2} \Delta S \geq \sum_{ijk} \left(h_{ijk}^a\right)^2 + \left(\frac{(1-2n-4p)}{n+2p}\right)S + \frac{c}{4} (n+1)S
\]

(2.15)

### 3.0 PROOF OF THEOREM

Let \( M \) be an \( n \)-dimensional anti-invariant maximal spacelike submanifold sometrically immersed in \( M^{n+p}_{n+p}(c), c \neq 0 \). Now assuming that \( M \) is compact and orientable, we have the integral formula

\[
0 \leq \int_M \sum_{ijk} \left(h_{ijk}^a\right)^2 *1 + \sum_M h_{ij}^a \Delta h_{ij}^a *1
\]

where *1 is the volume element of \( M \). From (2.15) we see that

\[
\sum_{ijk} \left(h_{ijk}^a\right)^2 - \frac{1}{2} \Delta S \leq \left(\frac{(2n+4p-1)}{n+2p}\right)S - \frac{c}{4} (n+1)S
\]

By a well known theorem of E. Hopf [3], \( \Delta S = 0 \) and thus we have

\[
0 \leq \int_M \left(\frac{(2n+4p-1)}{n+2p}\right)S - \frac{c}{4} (n+1)S *1
\]

(3.1)

Assume everywhere on \( M \). Then (3.1) implies that the second fundamental form of \( M \) is parallel and hence \( S \) is constant. Therefore, \( S = \frac{(n+1)(n+2p)}{4(2n+4p-1)c} \)

0 and \( M \) is totally geodesic or \( S > \frac{(n+1)(n+2p)}{4(2n+4p-1)c} \) at some point of \( M \).

As an immediate consequence of this result we get;

**Corollary 3.1**

Let \( M \) be an \( n \)-dimensional compact anti-invariant maximal spacelike submanifold

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of \( \overline{M}_{p}^{n+p}(c), c \neq 0 \). If the second fundamental form of \( M \) is parallel then \( M \) is totally geodesic.

4.0 CONCLUSION

In this paper, we studied the geometry of an \( n \)-dimensional anti-invariant maximal spacelike submanifold \( M \) immersed in an indefinite complex space form by computing the square of the length of the second fundamental form. In conclusion, we find

\[
S = \frac{(n+1)(n+2p)}{4(2n+4p-1)} \cdot c
\]

that either \( M \) is totally geodesic or

\[
S > \frac{(n+1)(n+2p)}{4(2n+4p-1)}c
\]

or at some point of \( M \), Moreover, if the second fundamental form of the submanifold is parallel then the submanifold is totally geodesic.
REFERENCES


