SOME RESULTS ON ANTI-INVARIANT MAXIMAL SPACELIKE SUBMANIFOLDS OF AN INDEFINITE COMPLEX SPACE FORM

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ABSTRACT
This paper looks into the geometry of an n-dimensional anti-invariant maximal spacelike submanifold M immersed in an indefinite complex space form $\bar{M}(c), c \neq 0$. Let M be an n-dimensional compact anti-invariant maximal spacelike submanifold of $\bar{M}_p^{n+p}(c), c \neq 0$. Then we show that either M is totally geodesic or $S = \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ or at some point of M, $S > \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$.

Key words: Anti-invariant submanifold, complex space form

1.0 INTRODUCTION

Among all submanifolds of a Kaehler manifold, there are two classes the class of anti-invariant submanifolds and that of holomorphic submanifolds. A submanifold of a Kaehler manifold is called an anti-invariant (resp. holomorphic) if each tangent space of the submanifold is mapped into the normal space (resp. itself) by the almost complex structure of the Kaehler manifold (Chen *et al*., 1974). A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form. Let $\bar{M}(c), c \neq 0$ be an indefinite complex space form of holomorphic sectional curvature c, complex dimension (n+p), $p \neq 0$ and index 2p. Let M be an n-dimensional anti-invariant maximal spacelike submanifold isometrically immersed in . We call M a spacelike submanifold if the induced metric on M from that of the ambient space is positive definite. Let J be the almost complex structure of . An n-dimensional Riemannian manifold M isometrically immersed in is called an anti-invariant submanifold of if each tangent space of M is mapped into the normal space by the almost complex structure J. Let h be the second fundamental form of M in and denote by S the square of the length of the second fundamental form h.

Our main result is:

Theorem.

Let M be an n-dimensional compact anti-invariant maximal spacelike submanifold of $\bar{M}_p^{n+p}(c), c \neq 0$. Then either M is totally geodesic or $S = \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ or at some point of M, $S = \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$

2.0 LOCAL FORMULAE

We choose a local field of orthonormal frames;

$$\left\{e_1,...,e_n; e_{n+1},...,e_{n+p}; e_{1^*} = Je_1,...,e_{n^*} = Je_n; e_{(n+1)^*} = Je_{n+1},...,e_{(n+p)^*} = Je_{n+p}\right\}$$

in $\bar{M}_p^{n+p}(c)$ such that restricted to M, the vectors $\{e_1,...,e_n\}$ are tangent to M and the rest are normal to M. With respect to this frame field of, $\bar{M}_p^{n+p}(c)$ let $w^1,...,w^n; w^{n+1},...,w^{n+p}; w^{1^*},...,w^{n^*}; w^{(n+1)^*},...,w^{(n+p)^*}$ be the field of dual frames. Unless otherwise stated, we shall make use of the following convention on the ranges of indices: $1 \leq A,B,C,D \leq n+p;\ 1 \leq i,j,k,l,m \leq n;\ \ n+1 \leq a,b,c \leq n+p;$ and when a letter appears in any term as a subscript and a superscript, it is understood that this letter is summed over its range. Besides

, $e_i = g(e_i, e_i) = g(Je_i, Je_i) = 1$, when $1 \leq i \leq n$,

.

$\varepsilon_a = g\left(e_a, e_a\right) = g\left(Je_a, Je_a\right) = -1$ when $n+1 \le a \le n+p$.

Then the structure equations of $\bar{M}_p^{n+p}(c), c \neq 0$ are;

$d\omega^A + \sum \varepsilon_B \omega_B^A \wedge \omega^B = 0$, $\omega_B^A + \omega_A^B = 0$, $\omega_j^i = \omega_{j*}^{i*}$, $\omega_j^{i*} = \omega_i^{j*}$,

$$d\omega_B^A + \sum_C \varepsilon_C \omega_C^A \wedge \omega_B^C = \frac{1}{2}\sum_{CD} \varepsilon_C \varepsilon_D \bar{R}_{BCD}^A \omega^C \wedge \omega^D,$$

$$\bar{R}_{BCD}^A = \frac{c}{4}\varepsilon_C \varepsilon_D \left(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC} + J_{AC}J_{BD} - J_{AD}J_{BC} + 2J_{AB}J_{CD}\right)$$

where $\bar{R}_{BCD}^A$ denote the components of the curvature tensor $\bar{R}$ on. Restricting these forms to M we have;

$\omega^a = 0,$ $\omega_i^a = \sum_i h_{ij}^a \omega^i,$ $h_{ij}^a = h_{ji}^a$, $d\omega^i = -\sum \omega_j^i \wedge \omega^j$,

$\omega_j^i + \omega_i^j = 0$, $d\omega_j^i = -\sum \omega_k^i \wedge \omega_j^k + \frac{1}{2}\sum_{kl} R_{jkl}^i \omega^k \wedge \omega^l,$

$R_{jkl}^i = \bar{R}_{jkl}^i - \sum_a \left(h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a\right)$, $d\omega^a = -\sum_b \omega_b^a \wedge \omega_b,$

$d\omega_b^a = -\sum_c \omega_c^a \wedge \omega_b^c + \frac{1}{2} R_{bij}^a \omega^i \wedge \omega^j,$

$$R_{bij}^a = \sum_k \left(h_{ik}^a h_{kj}^b - h_{kj}^a h_{ki}^b\right) \quad \text{...(2.1)}$$

From the condition on the dimensions of M and $\bar{M}_p^{n+p}(c)$ it follows that $e_{1*}, \ldots, e_{n*}$ is a frame for $T^\perp(M)$. Noticing this, we see that

$$R_{jkl}^i = \frac{c}{4}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) - \sum_a \left(h_{ik}^a h_{jl}^a - h_{il}^a h_{jk}^a\right) \quad \text{...(2.2)}$$

We call $H = \frac{1}{n}\sqrt{\sum_a \left(\sum_i h_{ii}^a\right)^2}$ the mean curvature of M and $S = \sum_{ija}\left(h_{ij}^a\right)^2$ the square of the length of the second fundamental form. If H is identically zero, then M is said to be maximal. M is totally geodesic if h = 0.

From (2.2), we have the Ricci tensor R_j^i given by

$$R_j^i = \sum_k R_{kjk}^i = \frac{(n-1)}{4}c\delta_{ij} + \sum_{ak} h_{ik}^a h_{kj}^a \quad \text{..(2.3)}$$

Thus the Ricci curvature R is;

$$R = R_i^i = \frac{c}{4}(n-1) + S \qquad \text{...(2.4)}$$

From (2.3) the scalar curvature r is given by

$$\mathrm{r} = \sum_j R_j^j = \frac{n(n-1)}{4}c + S \qquad \text{...(2.5)}$$

Let h_{ijk}^a denote the covariant derivative of h_{ij}^a. Then we define h_{ijk}^a by

$$\sum_k h_{ijk}^a \mathrm{w}^k = dh_{ij}^a + \sum_k h_{kj}^a \mathrm{w}_i^k + \sum_k h_{ik}^a \mathrm{w}_j^k + \sum_b h_{ij}^b \mathrm{w}_b^a \qquad \text{...(2.6)}$$

and $h_{ijk}^a = h_{ikj}^a$. Taking the exterior derivative of (2.6) we define the second covariant derivative of h_{ij}^a by

$$\sum_l h_{ijkl}^a \mathrm{w}^l = dh_{ijk}^a + \sum_l h_{ljk}^a \mathrm{w}_i^l + \sum_l h_{ilk}^a \mathrm{w}_j^l + \sum_l h_{ijl}^a \mathrm{w}_k^l + \sum_b h_{ijk}^b \mathrm{w}_b^a \qquad \text{...........................}$$

(2.7) Using (2.7), we obtain the Ricci formula;

$$h_{ijkl}^a - h_{ijlk}^a = \sum_m h_{mj}^a R_{ikl}^m + \sum_m h_{im}^a R_{jkl}^m + \sum_b h_{ij}^b R_{bkl}^a \qquad \text{..(2.8)}$$

The Laplacian of the second fundamental form is defined as $\Delta h_{ij}^a = \sum_l h_{ijll}^a$. Therefore,

$$\Delta h_{ij}^a = nH_{ij} + \frac{c}{4}(n+1)\sum h_{ij}^a - \sum_{bmk} h_{mi}^a h_{mk}^b h_{kj}^b + \sum_{bmk} h_{mi}^a h_{mj}^b h_{kk}^b - \sum_{bmk} h_{km}^a h_{mj}^b h_{ik}^b$$
$$+ \sum_{bmk} h_{km}^a h_{mk}^b h_{ij}^b + \sum_{bmk} h_{ki}^b h_{jm}^a h_{mk}^b - \sum_{bmk} h_{ki}^b h_{mk}^a h_{mj}^b \qquad \text{...........}$$

(2.9)

where H_{ij} is the second covariant derivative of H.

For M maximal in $\bar{M}_p^{n+p}(c)$, (2.9) becomes,

$$\Delta h_{ij}^a = \frac{c}{4}(n+1)\sum h_{ij}^a - \sum_{bmk} h_{mi}^a h_{mk}^b h_{kj}^b - \sum_{bmk} h_{km}^a h_{mj}^b h_{ik}^b + \sum_{bmk} h_{km}^a h_{mk}^b h_{ij}^b$$
$$+ \sum_{bmk} h_{ki}^b h_{jm}^a h_{mk}^b - \sum_{bmk} h_{ki}^b h_{mk}^a h_{mj}^b \qquad \text{................}$$

(2.10)

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From $\frac{1}{2}\Delta\sum_{aij}\left(h_{ij}^{a}\right)^{2}=\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}+\sum_{aij}h_{ij}^{a}\Delta h_{ij}^{a}$ we obtain,

$$\frac{1}{2}\Delta\sum_{aij}\left(h_{ij}^{a}\right)^{2}=\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}+\frac{c}{4}(n+1)\sum_{aij}\left(h_{ij}^{a}\right)^{2}-\sum_{abijkl}h_{ij}^{a}h_{kl}^{a}h_{lk}^{b}h_{ij}^{b}$$

$$+\sum_{abijkl}\left(h_{li}^{a}h_{lj}^{b}-h_{li}^{b}h_{lj}^{a}\right)\left(h_{ki}^{a}h_{kj}^{b}-h_{ki}^{b}h_{kj}^{a}\right) \quad \text{...(2.11)}$$

For each a let H_a denote the symmetric matrix $\left(h_{ij}^{a}\right)$. Then (2.11) can be written as

$$\frac{1}{2}\Delta\sum_{aij}\left(h_{ij}^{a}\right)^{2}=\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}+\frac{c}{4}(n+1)\sum_{aij}\left(h_{ij}^{a}\right)^{2}-\sum_{ab}\left(trH_aH_b\right)^{2}$$

$$+\sum_{ab}tr\left(H_aH_b-H_bH_a\right)^{2} \quad \text{...(2.12)}$$

where trH_aH_b denotes the trace of the matrix H_aH_b.

In the sequel, we need the following lemma proved in (Chern *et al.*, 1970) by S. S. Chern, M. do Carmo and S. Kobayashi.

Lemma 2.1:

Let A and B be symmetric nxn-matrices. Then, $-Tr\left(AB-BA\right)^{2}\leq 2TrA^{2}TrB^{2}$ and equality holds for non-zero matrices A and B if and only if A and B can be transformed by an orthogonal matrix simultaneously into scalar multiples of $\overline{A}$ and $\overline{B}$ respectively, where

$$\overline{A}=\left(\begin{array}{cc|c}0 & 1 & 0\\ 1 & 0 & \\ \hline & 0 & 0\\ & & \end{array}\right), \qquad \overline{B}=\left(\begin{array}{cc|c}1 & 0 & 0\\ 0 & -1 & \\ \hline & 0 & 0\\ & & \end{array}\right).$$

Moreover, if A_1, A_2, A_3 are three symmetric nxn-matrices such that $-Tr\left(A_aA_b-A_bA_a\right)^{2}=2TrA_a^{2}TrA_b^{2},\ 1\leq a,b\leq 3,\ a\neq b,$ then at least one of the matrices A_a must be zero.

Let $S_{ab}=\sum_{abij}h_{ij}^{a}h_{ij}^{b}$. Then (n+2p)×(n+2p)-matrix $\left(S_{ab}\right)$ is symmetric and can be assumed to be diagonal for a suitable choice of $e_{n+1},\ldots,e_{n+p}$. Setting $S_a=S_{aa}=trH_a^{2}$ and $S=\sum_{a}S_a$, equation (2.12) reduces to

$$\frac{1}{2}\Delta S=\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}+\frac{c}{4}(n+1)S-\sum_{ab}\left(trH_{a}H_{b}\right)^{2}+\sum_{ab}tr\left(H_{a}H_{b}-H_{b}H_{a}\right)^{2}$$ ....

(2.13) On the other hand, using Lemma 2.1 we have,

$$\frac{c}{4}(n+1)S-\sum_{ab}\left(trH_{a}H_{b}\right)^{2}+\sum_{ab}tr\left(H_{a}H_{b}-H_{b}H_{a}\right)^{2}\geq\frac{c}{4}(n+1)S-\sum_{a}S_{a}^{2}-2\sum_{ab}S_{a}S_{b}$$

$$=\left(\frac{(1-2n-4p)}{n+2p}S+\frac{c}{4}(n+1)\right)S+\frac{1}{(n+2p)}\sum_{a>b}\left(S_{a}-S_{b}\right)^{2}$$ ....

(2.14)
which, together with (2.13), implies that

$$\frac{1}{2}\Delta S\geq\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}+\left(\frac{(1-2n-4p)}{n+2p}S+\frac{c}{4}(n+1)\right)S$$ ....(2.15)

3.0 PROOF OF THEOREM

Let M be an n-dimensional anti-invariant maximal spacelike submanifold sometrically immersed in $\bar{M}_{p}^{n+p}(c), c\neq 0$. Now assuming that M is compact and orientable, we have the integral formula $0\leq\int_{M}\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}*1=-\int_{M}\sum_{aij}h_{ij}^{a}\Delta h_{ij}^{a}*1$, where *1 is the volume element of M. From (2.15) we see that $\sum_{aijk}\left(h_{ijk}^{a}\right)^{2}-\frac{1}{2}\Delta S\leq\left(\frac{(2n+4p-1)}{n+2p}S-\frac{c}{4}(n+1)\right)S$.

By a well known theorem of E. Hopf [3], $\Delta S=0$ and thus we have

$$0\leq\int_{M}\left(\frac{(2n+4p-1)}{n+2p}S-\frac{c}{4}(n+1)\right)S*1$$ ....(3.1)

Assume $S\leq\frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ everywhere on M. Then (3.1) implies that the second fundamental form of M is parallel and hence S is constant. Therefore, S = 0 and M is totally geodesic or $S=\frac{(n+1)(n+2p)}{4(2n+4p-1)}c$. Except for these two cases, $S>\frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ at some point of M.

As an immediate consequence of this result we get;

Corollary 3.1

Let M be an n-dimensional compact anti-invariant maximal spacelike submanifold of $\bar{M}_p^{n+p}(c), c \neq 0$. If the second fundamental form of M is parallel then M is totally geodesic.

4.0 CONCLUSION

In this paper, we studied the geometry of an n-dimensional anti-invariant maximal spacelike submanifold M immersed in an indefinite complex space form by computing the square of the length of the second fundamental form. In conclusion, we find that either M is totally geodesic or $S = \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$ or at some point of M, $S > \frac{(n+1)(n+2p)}{4(2n+4p-1)}c$. Moreover, if the second fundamental form of the submanifold is parallel then the submanifold is totally geodesic.

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