A NEW THIRD ORDER ROTATABLE DESIGN IN FIVE DIMENSIONS THROUGH BALANCED INCOMPLETE BLOCK DESIGNS

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Abstract
Response surface methodology (RSM) is a collection of statistical and mathematical techniques useful for developing, improving and optimising processes. To cut on costs, an experimenter has to make a choice of the experimental design prior to experimentation. The most extensive applications of RSM are in the particular situations where several input variables potentially influence some performance measure or quality characteristic of the process. The field of RSM consists of the experimental strategy for exploring the space of the process or independent variables and empirical statistical modeling to develop an appropriate relationship between the yield and the process variables and optimisation methods for finding the values of the process variables that produce desirable values of the response. The fitting of the response surface can be complex and tedious if done haphazardly. The use of rotatable designs has been suggested. These designs ensure equal precision on the response estimates. The advent of high speed computers, the realisation of the importance and need to choose and adopt an experimental design that is best according to some well defined statistical criterion, led to the development of a subject like optimality of designs. In view of the massive research effort in improving the statistical tools for investigation of response surfaces, it would be hoped that experimenters would be increasingly using the sophisticated developed statistical tools, which is not the case in general terms. Experiments of this kind could be widely applicable in human medicine, veterinary medicine, agriculture and in general, product research-innovation development for optimum resource utilisation based industrialisation process in line with the Kenya Vision 2030. In this paper, a third order rotatable design in five dimensions from a third order rotatable design in lower dimensions is constructed through balanced incomplete block designs (BIBDS). The result of the experiment in lower dimensional design need not be discarded. RSM will be appropriate to the study of phenomena that are presently not well understood to permit the mechanistic approach where the mechanistic approach is itself used when the mechanisms of some scientific phenomena are understood sufficiently well.

Key words: Response surface, rotatable designs, third order

1.0 Introduction

Draper (1960c) provided a method of constructing second-order rotatable designs in k-dimensions from the second-order rotatable designs in $(k-l)$-dimensions. Herzberg (1967) provided an alternative method which always works and for which the results of the experiments performed according to the $(k-l)$-dimensional design need not be discarded. The method of construction of a third order rotatable design presented here shares some of the features of Herzberg's method, and is analogous to the method for second-order rotatable designs suggested in Huda (1981). For example, the experimenter may start with a $(k-l)$-dimensional design. If after performing the design, either singular or non-singular, the experimenter feels that one more factor should be included, then he can proceed by the method described. For some new sequential methods, the reader is referred to Huda (1982b, 1983) and Mutiso and Koske (2005, 2006).

Box and Hunter (1957) introduced rotatable designs for the exploration of response surfaces. For these designs, the variance of the estimated response is constant at points equidistant from the centre of the designs. The k-dimensional point set $(x_{1u},..., x_{ku})$ $(u = 1,..., N)$ is a third order rotatable arrangement if,

$$\sum_{u=1}^{N} x_{iu}^2 = A \qquad (i = 1,...., k)$$

$$\sum_{u=1}^{N} x_{iu}^4 = 3\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 = 3B \qquad (i \neq j; \qquad i, j = 1,..., k) \quad \text{.................................(1)}$$

$$\sum_{u=1}^{N} x_{iu}^6 = 5\sum_{u=1}^{N} x_{iu}^4 x_{ju}^2 = 15\sum_{u=1}^{N} x_{iu}^2 x_{ju}^2 x_{lu}^2 = 15C \quad (i \neq j \neq l; \quad i, j, l = 1,...,k),$$

and all other sums of powers and products up to order six are zero.

From Gardiner, Grandage *et al.* (1959), it is known that if (1) is satisfied, then the design points also satisfy,

$$\frac{NB}{A^2} \geq \frac{k}{(k+2)} \quad \text{.......... (2)}$$

$$\frac{AC}{B^2} \geq \frac{(k+2)}{(k+4)} \quad \text{..........(3)}$$

and the arrangement forms a non-singular third order design if and only if strict inequality is achieved in (2) and (3). The strict inequality in (2) can always be achieved, if necessary by the addition of centre points. It is known (Draper, 1960a) that the strict inequality in (3) is achieved by a third-order rotatable arrangement if and only if the points lie on two or more distinct spheres centered at the origin and then the strict inequality in (2) is also automatically obtained.

2.0 Construction of the Design

Huda (1987) gave a method of constructing designs for $k-l$ dimensional points $\{x_{1u},...,x_{k-l,u}\}$ $(u = 1,2,..., N)$ which satisfy (1) in $k-l$ dimensional space to k dimensional points.

The problem is to derive a 5-dimensional third order rotatable design such that the points $(x_{1u}, x_{2u}, 0,0,0)$ $(u = 1, 2,..., N))$ are included in the design. This method entails that a B.I.B is utilized in the construction of the designs. Let $B.I.B(t,b,r,s,\lambda,\mu)$ denote a doubly balanced incomplete design of t treatments in b blocks each of size s with r, λ and μ representing the number of occurrences of each treatment, each pair of treatments and each triplet of treatments, respectively. The main task of this paper is to construct a new design of five dimensions.

Now consider the double balanced incomplete block designs with $t = k = 5$ and $s = k - l = 2$. It is assumed that the first block $h_i = a(i = 1,2)$ and $h_i = 0(i = 3,4,5)$ .Then for each block of $B.I.B(5,10,4,2,1,0)$ and each $u(u = 1,2,..., N')$ there is an associated point generated by replacing the $(k - l)$ nonzero entries of $(h_1,..., h_k)$ by $x_{1u},...,x_{k-l,u}$ in any order without replacing the x_{iu}'s $(i = 1,2)$. let the x_{iu}'s be placed in the ascending order of the i's in the first block. Let $ABIB(5,10,4,2,1,0)$ denote the set of all such points generated from the corresponding block design and the given $(k - l)-$ dimensional arrangement. Then by combining $ABIB(5,10,4,2,1,0)$ with symmetric point sets it is possible to obtain a design of five dimensions satisfying (1), (2), (3) and containing $(x_{1u}, x_{2u}, 0,0,0)$ $(u = 1,2,..., N')$.

3.0 The Design

Consider a third order rotatable design in two dimensions consisting of N' points equally spaced on a circle of radius ρ. Then from Bose and Cater (1959) and Gardiner *et al.*(1959) it is known that for this arrangement,

$$A = \frac{N'}{2}\rho^2, B = \frac{N'}{8}\rho^4, \text{ and } C = \frac{N'}{48}\rho^6.$$

A, B, C are as defined in (1). Consider the 5-dimensional point set $BIB(5,10,4,2,1,0)$ generated from this arrangement. Combine the $5N'$ points of this set with the 132 points of S(a,a,a,0,0), S(b,b,b,b,b) and S(d,d,d,d,d) with $d^2 = b^2 t, t \geq 0$. This arrangement forms a third rotatable design in 5-dimensions if;

$$a^2 = \left(\frac{13}{24}C\right)^{\frac{1}{3}}, \qquad b^2 = \left[\left(\frac{3C}{64}\right)\frac{1}{1+4t^3}\right]^{\frac{1}{3}}$$

and if t is such that,

$$\frac{(1+4t^2)^3}{(1+4t^3)^2} = \frac{\left(9B - 24\left(\frac{13C}{24}\right)^{\frac{2}{3}}\right)^3}{64(9C^2)}$$

If N $'$ = 8, then t is needed such that

$$\frac{(1+4t^2)^3}{(1+4t^3)^2} = 4.53255$$ which gives t = 0.6781. Hence a third order rotatable

design in 5-dimensions with 320 points is obtained.
The design is given by

S(a,a,a,0,0), S(b,b,b,b,b), $S(\frac{\rho}{\sqrt{2}}, \frac{\rho}{\sqrt{2}}, 0,0,0)$ and 4 sets of S(d,d,d,d,d) and S($\rho, 0,0,00)$ each, where $a^2 = 0.4486\,\rho^2, b^2 = 0.1515\,\rho^2$ and $d^2 = 0.1027\,\rho^2$.

4.0 Results and Conclusions

A third order rotatable design in five dimensions with 320 points is obtained through balanced incomplete block designs. Therefore, a five-dimensional design does exist as we have shown in this paper.

Many third order rotatable designs have been described in Gardiner *et al.* (1959); Draper (1960a, b, 1961); Thaker and Das (1961); Das and Narasimham (1962), Herzberg (1964); Tyagi (1964), and Nigam (1967). These designs would usually require many more points than the available minimal point designs and hence may not always be desirable. For example, the experimenter might also be interested in some of the ($k - l$) subsets of the k factors. These subsets may be identified with the blocks generating ABIB (t= k ,b,r, $k - l, \lambda, \mu)$ so that the k-dimensional third order rotatable design containts third order rotatable designs in ($k - l$)- dimenstions involving the subsets of factors the experimenter is interested in.

During last few decades, rotatable designs using BIBDs have been introduced by a number of workers. This introduces a new method of constructing higher level of third order rotatable designs using BIBDS.

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References

Bose R. C. and Carter R. L. (1959). Complex representation in the construction of rotatable designs. *Ann. Math. Statist.,* **30**, pp 771-80

Box G. E. P. and Hunter J. S. (1957). Multifactor experimental designs for exploring response surfaces*. Ann. Math. Statist.,* **28**, pp 195-241.

Das M. N. and Narasimham V. L. (1962). Construction of rotatable designs through balanced incomplete block designs. *Ann. Math. Statist.,* **33**, pp 1421-39.

Draper N. R. (1960a). Third-rotatable designs in three dimensions. *Ann. Math. Statist.,* **31**, pp 866-74.

Draper N. R. (1960b). A third-order rotatable designs in four dimensions. *Ann. Math. Statist.,* **31**, pp 875-77.

Draper N. R. (1960c). Second-order rotable designs in four and more dimensions*. Ann. Math. Statist.,* **31**, pp 23-33.

Draper N. R. (1961). Third-order rotatable Designs in three dimensions: Some specific designs. *Ann. Math. Statist.,***32**, pp 910-13.

Gardiner D. A., Grandage A. H. E. and Hader, R. J. (1959). Third-order rotatable designs for exploring response surfaces. *Ann. Math. Statist.,***30**, pp 1082-1096.

Herzberg, A. M. (1964). Two third-order rotatable designs in four dimensions *Ann. Math. Statist***., 35**, pp 445-46.

Herzberg A. M. (1967). A method for the construction of second-order rotatable designs in k dimensions. *Ann. Math. Statist.,* **38**, pp 177-80.

Huda S. (1981). A method of constructing second-order rotatable designs. *Calcutta Statist. Assoc. Bull.,* **30,** pp 139-44.

Huda S. (1982a). Some third-order rotatable designs. Biom. J**., 24,** pp 257-63.

Huda S. (1982b). Some third-order rotatable designs in three dimensions. *Ann. Inst. Statist., Math A.*, **34,** pp365-71.

Huda S. (1983). Two third-order rotatable designs infour dimensionsJ. *Statist. Plan. Inf.,* **8,** 241-43**.**

Huda S. (1987). The construction of third-order rotatable designs in K dimensions from those in lower dimensions. *Pak. J. Statist*., **3**, pp 11-16.

Mutiso J. M. and Koske J. K. (2005). Some third order rotatable designs in five dimensions. *East African Journal of Statistics*, **1**, pp 117-122.

Mutiso J. M. and Koske J. K. (2006). Some third order rotatable designs in six dimensions. Accepted for publication in the journal of Agriculture, science and Technology(JAST).

Nigam A. K. (1967). On third-order rotatable designs with smaller numbers of levels. J. *Ind. Soc. Agric. Statist.,* **19,** pp 36-41.

Thaker P. J. and Das M. N. (1961). Sequential third order rotatable designs up to eleven factors. *J. Ind. Soc. Agriculture. Statist.,* **13**, pp 218-231.

Tyagi B. N (1964). On construction of second and third order rotatable designs through pairwise balanced and doubly balanced designs. *Calcutta Statist. Assoc. Bull.*, **13**, pp 150-162.

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