PARAMETRIC CHANGE POINT ESTIMATION, TESTING AND CONFIDENCE INTERVAL APPLIED IN BUSINESS

A. W. Gichuhi[1], J. Franke[2] and J. M. Kihoro[1]
[1], Jomo Kenyatta University of Agriculture and Technology
[2]University of Kaiserslautern, Kaiserslautern, Germany
Email: agwaititu@yahoo.com

Abstract
In many applications like finance, industry and medicine, it is important to consider that the model parameters may undergo changes at unknown moment in time. This paper deals with estimation, testing and confidence interval of a change point for a univariate variable which is assumed to be normally distributed. To detect a possible change point, we use a Schwarz Information Criterion (SIC) statistic whose asymptotic distribution under the null hypothesis is determined. The percentile bootstrap method is used to construct the confidence interval of the estimated change point. The developed tools and methods are applied to the 1987 – 1988 US trade deficit data. Our results show that a significant change in US trade deficit occurred in November 1987. Further, it is shown that the percentile bootstrap confidence intervals are not always symmetrical.

Key words: Change point, Schwarz information criterion, percentile bootstrap

1.0 Introduction

In many applications of statistics such as the financial, industrial and medical fields, it is important to consider that the model parameters may undergo changes at an unknown moment of time. The time moment when the model has changed is called the change point. Other synonyms are probabilistic diagnostics and disorder problems.

The change point problem is twofold: Change point detection and change point estimation. Depending on whether the probabilistic model of data is known or not, one can distinguish between parametric, semi-parametric and non-parametric methods of change point detection and estimation. Worsley (1983) used the likelihood ratio method to test for a change in probability of a sequence of independent binomial variables.

Non-parametric detection of a change point in a sequence of random variables was studied by many authors. Page (1955) used the cumulative sum technique to test for a possible change point. Worsley (1983) used the cumulativesum statistics to test for a change in probability of a sequence of independentbinomial random variables.

Page's CUSUM and Shewhart's control chart are some of the popular procedures used when both the pre-change distribution f_0 and post-change distribution f_1 are completely specified. Yashchin (1997) uses the likelihood ratio strategy. However, in line with statistical quality control, standard procedures assume that the pre-change distribution f_0 is known but the post-change distribution f_1 is unknown and therefore has to be estimated. Such a study has been done in Siegmund and Venkatraman (1995).

The maximum likelihood estimate (MLE) method has been used to estimate a change point when the probabilistic data model is known. Hinkley (1970) applied the MLE method to estimate a change point in a sequence of normally distributed random variables whereby he derived the asymptotic distribution of the estimator using random walk theory. Hinkley and Hinkley (1970) used the MLE method to estimate the change point in a sequence of zero-one variables.

Pettitt (1980) used a Mann-Whitney statistic to estimate a change point when it is known that a change has taken place at an unknown point in a sequence of random variables. In this work, the estimate is compared with MLE using Monte Carlo techniques and is found to be fairly constant over various distributions like normal distribution.

As indicated above, parametric test statistics for a change point are based on the likelihood ratio statistic and the estimation done using maximum likelihood method. More general results can be found in Csörgö, M., and Horvath, L.(1997).

This paper uses the theory of Schwarz Information Criterion (SIC) to detect and estimate a change point for a given sequence of normally distributed random variables.

2.0 Change Point Model Formulation

We assume that one is able to observe a sequence of independent normal observations whose distribution possibly changes from $N(\mu_b, \sigma_b)$ to $N(\mu_a, \sigma_a)$ at an unknown point in time, K .

That is

$$X = \begin{cases} X_i \sim N(\mu_b, \sigma_b) & , i = 1, 2, \ldots, K \\ X_i \sim N(\mu_a, \sigma_a) & , i = K+1, K+2, \ldots, n \end{cases}$$

2.1 Hypotheses

In this paper, the hypothesis of stability is defined as

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_n = \mu \text{ and } \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = \sigma^2 \qquad \ldots\ldots (1)$$

The alternative hypothesis is defined as

$$H_1 : \mu_1 = \ldots = \mu_K = \mu_b \neq \mu_{K+1} = \ldots = \mu_n = \mu_a \text{ and } \sigma_1^2 = \ldots = \sigma_K^2 = \sigma_b^2 \neq \sigma_{K+1}^2 = \ldots = \sigma_n^2 = \sigma_a^2 \qquad \ldots\ldots (2)$$

Under H_0, the mle's for μ and σ^2 are, respectively,

$$\hat{\mu} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \qquad \ldots\ldots (3)$$

and under H_1, the mle's for μ and σ^2 are, respectively,

$$\hat{\mu}_b = \bar{x}_b = \frac{1}{K}\sum_{i=1}^{K} x_i \quad \text{and} \quad \hat{\sigma}_a^2 = \frac{1}{K}\sum_{i=1}^{K}(x_i - \bar{x}_b)^2$$

$$\hat{\mu}_a = \bar{x}_a = \frac{1}{n-K}\sum_{i=K+1}^{n} x_i \quad \text{and} \quad \hat{\sigma}_a^2 = \frac{1}{n-K}\sum_{i=K+1}^{n}(x_i - \bar{x}_a)^2$$

2.2 Schwarz Information Criterions for the Change Point Inference

The Schwarz Information Criterion (SIC) was proposed by (Schwarz, 1978) and it is expressed as

$$SIC(m) = -2\log L(\hat{\Theta}_m) + m\log n, \quad m = 1, 2, \ldots, M \qquad \ldots\ldots (4)$$

Where m is the number of free parameters, $L(\hat{\Theta}_m)$ is the maximum likelihood function for model(*m*) and $m\log n$ is the penalty term

Using equation (4), the SIC under H_0 has 2 free parameters and it is clearly defined as

$$SIC(n) = n\log 2\pi + n\log\hat{\sigma}^2 + n + 2\log n \qquad \ldots\ldots (5)$$

where $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$ is the mle of σ^2 under H_0.

Similarly, using equation (4), the SIC under H_1 has 4 free parameters and it is defined as

$$SIC(K) = n\log 2\pi + K\log\hat{\sigma}_b^2 + (n-K)\log\hat{\sigma}_a^2 + n + 4\log n \quad \text{..........................} \quad (6)$$

for $2 \le K \le n-2$

where $\hat{\sigma}_b^2 = \frac{1}{K}\sum_{i=1}^{K}(x_i - \bar{x}_b)^2$ and $\bar{x}_b = \frac{1}{K}\sum_{i=1}^{K}x_i$ are the mle's of the variance and the mean,respectively, before the change point and

$\hat{\sigma}_a^2 = \frac{1}{n-K}\sum_{i=K+1}^{n}(x_i - \bar{x}_a)^2$ and $\bar{x}_a = \frac{1}{n-K}\sum_{i=K+1}^{n}x_i$ are the mle's of the variance and the mean, respectively, after the change point.

2.3 Change Point Estimation

As in Chen and Gupta (2000), we estimate the change point K by $\hat{K}$ such that

$$SIC(\hat{K}) = \min_{2\le K\le n-2} SIC(K) \quad \text{..........................} \quad (7)$$

2.4 Change Point Testing

If K is not fixed and is unknown, we follow the approach of Chen and Gupta (2000) and fail to reject H_0 iff

$$\Lambda_K = \min_{2\le K\le n-2}(SIC(K)\ ) + R_n(\alpha) - SIC(n) \quad \text{..........................} \quad (8)$$

is positive where $R_n(\alpha)$ is the critical value associated with the sample size n and significance level α.

2.4.1 Critical Values for SIC, $R_n(\alpha)$

Let $\Delta_{n,K} = \min_{2\le K\le n-K}\{SIC(K) - SIC(n)\ \}$ .... (9)

Then,

$$\begin{aligned}\Delta_{n,K} &= -\max_{2\le K\le n-K}\{SIC(K) - SIC(n)\ \} \\ &= -\max_{2\le K\le n-K}(K\log\hat{\sigma}_b^2 + (n-K)\log\hat{\sigma}_a^2 - n\log\hat{\sigma}^2 + 2\log n) \quad \text{.........}(10) \\ &= -\eta_{n,K}^2 + 2\log n\end{aligned}$$

where $$\eta_{n,K} = \left(\max_{2 \le K \le n-K} (K \log \hat{\sigma}_b^2 + (n-K)\log \hat{\sigma}_a^2 - n \log \hat{\sigma}^2) \right)^{\frac{1}{2}} \qquad \text{.......................(11)}$$

From equation (10), one has

$$\eta_{n,K} = \left(2 \log n - \Delta_{n,K}\right)^{\frac{1}{2}} \qquad \text{..(12)}$$

Theorem 1 Under H_0 , for all $x \in \mathbb{R}$, we have for

$$\eta_{n,K} = \left(2 \log n - \Delta_{n,K}\right)^{\frac{1}{2}}$$
$$\lim_{n \to \infty} P(a(\log n)\eta_{n,K} \le x + b \log n) = \exp\ (-2 \exp\ (-x)\ ) \qquad \text{.............................. (13)}$$

Where $a(\log n) = (2 \log\ \log n)^{\frac{1}{2}}$ and $b(\log n) = 2 \log \log n + \log \log \log n$
This result follows immediately from Theorem (2.1) of Gombay and Horvath (1996) and Theorem (2.27) of Chen and Gupta (2000).

Using equation (8) to determine the critical values, $R_n(\alpha)$**,** we note that under theorem (1),

$$\begin{aligned}
1-\alpha &= P(SIC(n) < \min_{2 \le K \le n-2} SIC(K) + R_n(\alpha) \mid H_0) \\
&= P(-\max_{2 \le K \le n-2} \{SIC(n) - SIC(K)\ \} > -R_n(\alpha) \mid H_0) \\
&= P(\Delta_n > -R_n(\alpha) \mid H_0) \\
&= P(-\eta_{n,K}^2 + 2 \log n > -R_n(\alpha) \mid H_0) \\
&= P(0 < \eta_{n,K} < (R_n(\alpha) + 2 \log n)^{0.5} \mid H_0) \\
&= P(-b(\log n) < a(\log n)\eta_{n,K} - b(\log n) < a(\log n) \quad (R_n(\alpha) + 2 \log n)^{0.5} - b(\log n) \mid H_0) \\
&\cong \exp\ \{-2 \exp\left(b(\log n) - a(\log n) \quad (R_n(\alpha) + 2 \log n)^{0.5}\right)\} - \exp\ \{-2 \exp\ (b(\log n)\ )\ \}
\end{aligned}$$
(14)

This then implies that
$$\exp\ \{-2 \exp\left(b(\log n) - a(\log n) \quad (R_n(\alpha) + 2 \log n)^{0.5}\right)\} \cong 1 - \alpha + \exp\ \{-2 \exp\ (b(\log n)\ )\ \}$$
So that
$$R_n(\alpha) \cong \left[-\frac{1}{a(\log n)} \log \log\ \{1 - \alpha + \exp\ (-2 \exp\ (b(\log n)\ )\ )\ \}^{-0.5} + \frac{b(\log n)}{a(\log n)} \right]^2 - 2 \log n$$
(15)

We computed $R_n(\alpha)$ for $\alpha = 0.10, 0.05, 0.025, 0.01$ under various sample sizes. The critical values are presented in table 1 below.

Table 1: Asymptotic critical values from equation (15), denoted as $R_n(\alpha)$

	$R_n(\alpha)$			
n α	**0.10**	**0.05**	**0.025**	**0.010**
7.	7.757992	12.909378	19.63085	35.69935
8.	7.404845	11.925257	17.23230	25.97584
9.	7.262061	11.540438	16.42328	23.94784
10.	7.168499	11.312834	15.99423	23.07060
11.	7.087391	11.138584	15.69148	22.52369
12.	7.010367	10.988932	15.44547	22.10831
13.	6.935751	10.854445	15.23288	21.76289
14.	6.863355	10.731205	15.04386	21.46347
15.	6.793235	10.617091	14.87308	21.19818
16.	6.725433	10.510699	14.71714	20.95987
17.	6.659935	10.410984	14.57361	20.74363
18.	6.596686	10.317120	14.44062	20.54582
19.	6.535604	10.228435	14.31671	20.36366
20.	6.476595	10.144368	14.20073	20.19494
21.	6.419556	10.064448	14.09171	20.03788
22.	6.364386	9.988275	13.98886	19.89103
23.	6.310986	9.915503	13.89152	19.75319
24.	6.259258	9.845834	13.79911	19.62336
25.	6.209112	9.779008	13.71117	19.50068
26.	6.160461	9.714797	13.62728	19.38444
27.	6.113227	9.652998	13.54708	19.27401
28.	6.067332	9.593433	13.47026	19.16885
29.	6.022706	9.535943	13.39655	19.06850
30.	5.979285	9.480385	13.32569	18.97255
40.	5.599685	9.007971	12.736662	18.19266
50.	5.293224	8.639973	12.291699	17.62215
60.	5.036173	8.338068	11.933873	17.17331
70.	4.814683	8.081879	11.634525	16.80384
80.	4.620012	7.859242	11.377170	16.49016
90.	4.446292	7.662302	11.151446	16.21778
100.	4.289397	7.485684	10.950411	15.97721
110.	4.146315	7.325548	10.769185	15.76186
120.	4.014778	7.179053	10.604207	15.56699
130.	3.893040	7.044036	10.452798	15.38910
140.	3.779721	6.918813	10.312891	15.22548
150.	3.673718	6.802049	10.182861	15.07403
160	3.574131	6.692662	10.061401	14.93309
170.	3.480216	6.589768	9.947450	14.80131

180.	3.391355	6.492633	9.840132	14.67758
190.	3.307024	6.400641	9.738717	14.56097
200.	3.226777	6.313270	9.642588	14.45073

3.0 Confidence Interval for the Change Point

Various methods for determining change point confidence intervals exist inliterature. One method involves the asymptotic distribution of $\hat{K} - K$, where K is the true change point and $\hat{K}$ is its estimate. See, for example, Hinkleyand Hinkley (1970) and Feder (1975).

Another approximation method involves the use of bootstrap methods.See for instance Hall (1992), Efon and Tibishirani (1993), Davison and Hinkley(1997) and Pastor-Barriuso *et al*(2003).

In this paper, we use the Percentile Bootstrap method to determine the confidence interval for the true change point K.

3.1 Percentile Bootstrap Confidence Intervalfor the Time of Change

We approximate the distribution of $\hat{K} - K$ using the percentilebootstrap techniqueas follows:

1. Given the original sample $X_i, i = 1,2,\ldots,n$,estimate the change point $\hat{K}_n$ and hence the MLEs $\bar{x}_b, \hat{\sigma}_b, \bar{x}_a, \hat{\sigma}_a$
2. Generate a bootstrap sample X_i^* such that

   $$X_i^* = N(1, \bar{x}_{bi}, \hat{\sigma}_{bi})|_{i \le \hat{K}_n} + N(1, \bar{x}_{ai}, \hat{\sigma}_{ai})|_{i \ge \hat{K}_n + 1}$$

   Where $N(f, g, h)$ is a normal variable of size f, mean g and standard deviation h.
3. Using the bootstrap sample $\{X_i^*\}_{i=1}^{n}$, replicate the estimated time of change, $\hat{K}_n^*$.
4. Repeat steps 2 and 3 B times. This step yields B independent bootstrap samples

$\{X_i^{*1}\}_{i=1}^{n}, \ldots, \{X_i^{*B}\}_{i=1}^{n}$ from which we get $\hat{K}_n^{*1}, \ldots, \hat{K}_n^{*B}$ bootstrap change points.

From these replicates, we are then able to estimate the distribution function of $\hat{K}_n^* - \hat{K}_n$

where $\hat{K}_n^*$ is the time of change estimate of the re-samples.

Proof

Suppose that $K^*_{\alpha/2}$ and $K^*_{1-\alpha/2}$ are the quartiles of K^*_n such that

$$P\left(K^*_n \le K^*_{\alpha/2}\right) = P\left(K^*_n > K^*_{1-\alpha/2}\right) = \alpha/2 \qquad (16)$$

One then has

$$P\left(K^*_{\alpha/2} \le K^*_n \le K^*_{1-\alpha/2}\right) = 1-\alpha \qquad (17)$$

Equation (16) implies that

$$P\left(K^*_{\alpha/2} - \hat{K}_n \le K^*_n - \hat{K}_n \le K^*_{1-\alpha/2} - \hat{K}_n\right) = 1-\alpha \qquad (18)$$

Assuming that we can approximate the quartiles of $\hat{K}_n - K$ by the quartiles of $K^*_n - \hat{K}_n$, we have

$$P\left(K^*_{\alpha/2} - \hat{K}_n \le \hat{K}_n - K \le K^*_{1-\alpha/2} - \hat{K}_n\right) \approx 1-\alpha \qquad (19)$$

So that

$$P\left(\hat{K}_n - \left(K^*_{1-\alpha/2} - \hat{K}_n\right) \le K \le \hat{K}_n - \left(K^*_{\alpha/2} - \hat{K}_n\right)\right) \approx 1-\alpha \qquad (20)$$

As noted in Efon and Tibshirani (1993), transforming equation (20) can give better a confidence interval. We therefore transform the random variable $\hat{K}_n$ using a symmetrical function say, $t(\,)$, denote as:

$$\hat{\omega}_n = t(\hat{K}_n) \qquad (21)$$

Using equations (20) and (21), one has

$$P\left(\hat{\omega}_n - \left(\omega^*_{1-\alpha/2} - \hat{\omega}_n\right) \le \omega \le \hat{\omega}_n - \left(\omega^*_{\alpha/2} - \hat{\omega}_n\right)\right) \approx 1-\alpha \qquad (22)$$

Due to symmetry, $\left(\omega^*_{1-\alpha/2} - \hat{\omega}_n\right) = -\left(\omega^*_{\alpha/2} - \hat{\omega}_n\right)$ so that equation (22) can be written as

$$P\left(\hat{\omega}_n + \left(\omega^*_{\alpha/2} - \hat{\omega}_n\right) \le \omega \le \hat{\omega}_n + \left(\omega^*_{1-\alpha/2} - \hat{\omega}_n\right)\right) \approx 1-\alpha \qquad (23)$$

So that

$$P\left(\omega^*_{\alpha/2} \le \omega \le \omega^*_{1-\alpha/2}\right) \approx 1-\alpha \qquad (24)$$

Transforming equation (24) back to the original scale gives

$$P\left(K^*_{\alpha/2} \le K \le K^*_{1-\alpha/2}\right) \approx 1-\alpha \qquad (25)$$

That is

$$P\left(K^*_{n,((B+1)\alpha/2)} \le K \le K^*_{n,((B+1)(1-\alpha/2))}\right) \approx 1-\alpha \qquad (26)$$

Therefore, the $\alpha -$percentile bootstrap confidence interval is given by:

$$\left(K^{*}_{n,((B+1)\alpha/2)}, K^{*}_{n,((B+1)(1-\alpha/2))}\right) \text{.........} \quad (27)$$

4.0 Empirical Results

4.1 The Data

Table (2) shows US trade deficit data from 1987 to 1988 in billions of dollars. The data is from Wheeler (1993).

Table 2: US Trade Deficits 1987-1988 ($ billions)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1987	10.7	13.0	11.4	11.5	12.5	14.1	14.8	14.1	12.6	16.0	11.7	10.6
1988	10.0	11.4	7.9	9.5	8.0	11.8	10.5	11.2	9.2	10.1	10.4	10.5

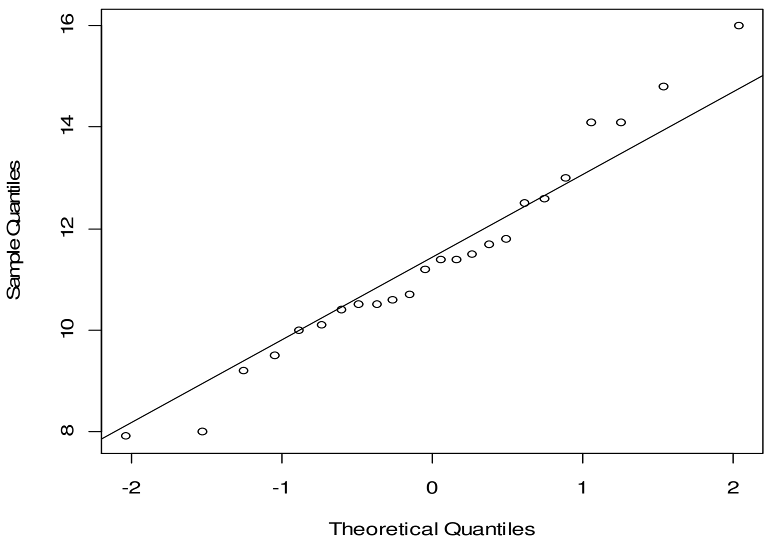

Figure 1: Normal Q-Q plot for the data in table 2

From figure 1, the US Trade Deficits from 1987-1988 were approximately normal in distribution.

4.2 Change Point Detection

Using equation (6), the $SIC(K)$ for $2 \le K \le n-2$ was computed and the results presented in figure 2. Using equation (7), the change point was detected at $\hat{K} = 11$. This implies that a change of deficits was detected to have occurred in November 1987.

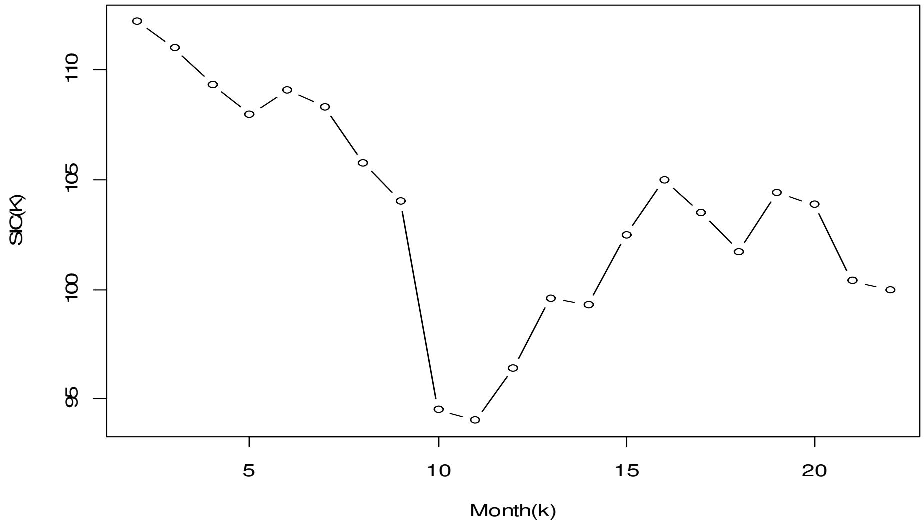

Figure 2: Change point detection for the US 1987-1988 trade deficits

This finding agrees with that of Taylor(2000) who analyzed the same data using a different change point tool and detected the change point to have occurred in November 1987.

4.2 Change Point Testing

Under section (2.4), equation (8), table 1 and figure 2, the following values were computed:

Table 3: $\min_{2 \le K \le 22} \{SIC(K)$, $R_{24}(\alpha)$ *and* $SIC(24)$ *for the1987-1988 US Trade Deficits*

$\min_{2 \le K \le 22} \{SIC(K)$	$R_{24}(\alpha)$				$SIC(24)$
	0.1	0.05	0.025	0.01	
94.02100	6.25926	9.84583	13.79911	19.62336	106.8370

From table 3 and using equation (8), we reject H_0 (equation(1)) at $\alpha = 0.05$. We therefore confirm that a change in mean and variance did actually occur in November 1987 at 5% significance level.

4.3 Confidence Interval for the Change Point

Figure 3 shows the bootstrap change points computed in line with section 3.1. 10,000 bootstraps for the change point were computed.

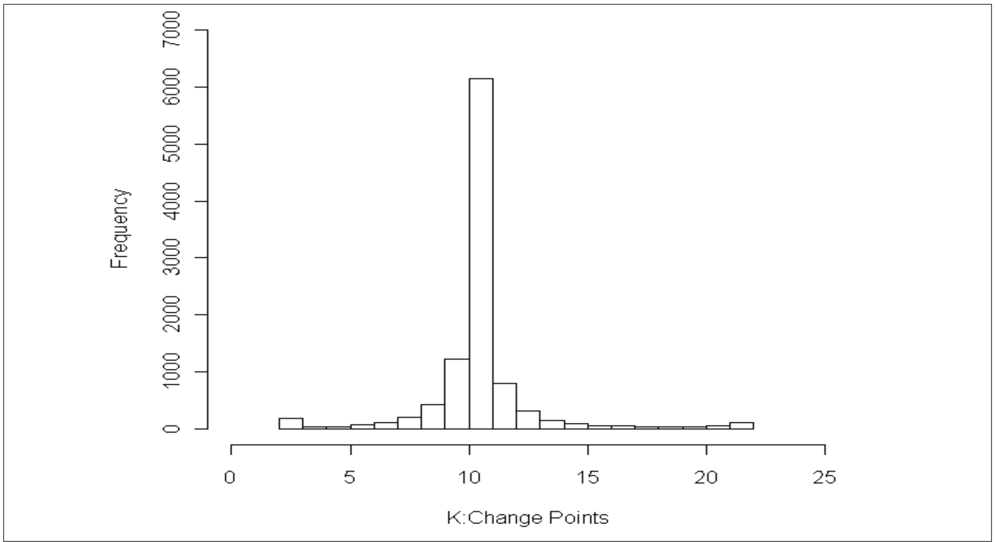

Figure 3: Histogram of bootstrap change points

Table 4: Confidence Interval results for B = 10000 bootstrap replications of the change point $\hat{K}_{24} = 11$ *for the 1987-1988 US Trade Deficits*

$\hat{K}_{24} = 11$	
Confidence Level	**Confidence Interval**
90%	8 - 14
95%	6 - 17

Our results indicate that the percentile bootstrap confidence intervals for the change point are not always symmetrical. These results are supported by Cook and Weisberg (1990).

5.0 Conclusion

In this paper, we have used the SIC criterion to detect a possible change point. This method has been applied to the 1987-1988 US Trade Deficits data and the results agreed with those of other authors. A table of critical values was computed for reference purposes.

The authors of this paper considered the case where only one change point was assumed to have occurred. More research should be done to model the case where two or more change points are assumed to have occurred.

References

Chen J. and Gupta A. K. (2000) *Parametric Change Point Analysis*, Birkhäuser, Boston

Cobb G. W. (1978). The problem of the Nile: conditional solution to a change point problem. *Biometrika,* **65(2)**, pp. 243-251.

Cook R. D. and Weisberg S. (1990). Confidence curves in nonlinear regression. *J. Amer. Statist. Assoc.* **85**(**410**), pp. 544 - 551.

Csörgö, M. and Horvath, L.(1997) *Limit Theorems in Change-Point Analysis*.Wiley, New York.

Davison A. C. and Hinkley D. V. (1997) *Bootstrap methods and their application*, 1 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge.

Efron, B. and Tibshirani, R. J. (1993) *An introduction to the bootstrap*, 57 of Monographs on Statistics and Applied Probability. Chapman and Hall.

Feder P. I. (1975). The log likelihood ratio in segmented regression. *Annals of Statistics*, **3**, pp. 84 – 97.

Gombay E. and Horvath L. (1996). On the rate of approximations for maximum likelihood tests in change-point models. *J. Multivariate Anal.* **56(1)**, pp. 120 - 152.

Hall P. (1992). *The bootstrap and Edgeworth expansion*. Springer Series in Statistics. Springer-Verlag, New York.

Hinkley D. V. (1970). Inference about the change-point in a sequence of random variables. *Biometrika*, **57**, pp. 1 - 17.

Hinkley D. V. and Hinkley E. A. (1970). Inference about the changepoint in a sequence of binomial variables. *Biometrika,* **57** , pp. 477 - 488.

Page E. S. (1955). A test for a change in a parameter occurring at an unknown point. *Biometrika,* **42**, pp. 523 - 527.

Pastor-Barriuso R. Guallar E. and Coresh J. (2003). Transition models for change-point estimation in logistic regression. *Statist. Med.,* **22** , pp. 1141 - 1162.

Pettitt A. N. (1980). A simple cumulative sum type statistic for the changepoint problem with zero-one observations. *Biometrika*, **67**(**1)**, pp. 79 - 84.

Schwarz G. (1978). Estimating the dimension of a model. *Annals of Statistics*, **6,** pp. 461 - 464.

Siegmund D. and Venkatraman E. S. (1995). Using the generalized likelihood ratio statistic for sequential detection of a change-point. *Annals of Statistics,* **23**(**1)**, pp. 255 - 271.

Taylor Wayne A. (2000), Change Point Analysis: A Powerful New Tool For Detecting Changes, WEB: http://www.variation.com/cpa/tech/changepoint.html.

Wheeler D. (1993). *Understanding Variation – The Key to Managing Chaos*, SPC Press, Knoxville, Tennessee.

Worsley K. J (1983). The power of likelihood ratio and cumulative sum tests for a change in a binomial probability. *Biometrika,* **70(2)**, pp. 455 - 464.

Yashchin, E.(1997) . Change-point models in industrial applications. In Proceedings of the Second World Congress of Nonlinear Analysts, Part 7 **30**, pp. 3997- 4006.